

ON REPRESENTATION OF INTEGERS BY BINARY QUADRATIC FORMS

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ABSTRACT. Given a negative $D > -(\log X)^{\log 2 - \delta}$, we give a new upper bound on the number of square free integers $< X$ which are represented by some but not all forms of the genus of a primitive positive definite binary quadratic form f of discriminant D . We also give an analogous upper bound for square free integers of the form $q + a < X$ where q is prime and $a \in \mathbb{Z}$ is fixed. Combined with the $1/2$ -dimensional sieve of Iwaniec, this yields a lower bound on the number of such integers $q + a < X$ represented by a binary quadratic form of discriminant D , where D is allowed to grow with X as above. An immediate consequence of this, coming from recent work of the authors in [BF], is a lower bound on the number of primes which come up as curvatures in a given primitive integer Apollonian circle packing.

§0. Introduction

Let $f(x, y) = ax^2 + bxy + cy^2 \in \mathbb{Z}[x, y]$ be a primitive positive-definite binary quadratic form of negative discriminant $D = b^2 - 4ac$. For $X \rightarrow \infty$, we denote by $U_f(X)$ the number of positive integers at most X that are representable by f . The problem of understanding the behavior of $U_f(X)$ when D is not fixed, i.e. $|D|$ may grow with X , has been addressed in several recent papers, in particular in [Bl] and [B-G]. What is shown in these papers, on a crude level, is that there are basically three ranges (we restrict ourselves to discriminants satisfying $\log |D| \leq O(\log \log X)$)

$$(i) \quad |D| \ll (\log X)^{(\log 2) - \varepsilon}. \text{ Then } U_f(X) \gg_{\varepsilon} X (\log X)^{-\frac{1}{2} - \varepsilon} \quad (0.1)$$

$$(ii) \quad |D| \gg (\log X)^{2(\log 2) + \varepsilon}. \text{ Then } U_f(X) \asymp \frac{X}{\sqrt{|D|}} \quad (0.2)$$

(iii) The intermediate range.

As Blomer and Granville explain in [B-G], this transitional behavior is due to the interplay between the size h of the class group \mathcal{C} and the typical number of prime factors of an integer $n \sim X$. A precise elaboration of the underlying heuristics was kindly communicated by V. Blomer to the authors and is reproduced next. The number of integers $n < X$ with k prime factors p split in the quadratic number field (i.e. $(\frac{D}{p}) = 1$) is of the order

$$\frac{X}{\log X} \frac{1}{2^k} \frac{(\log \log X)^{k-1}}{(k-1)!}. \quad (0.3)$$

Note that summation of (0.3) over k gives $\frac{X}{\sqrt{\log X}}$, which is the number of integers at most X represented by some form of discriminant D .

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Moreover, applying Stirling's formula, we see that the main contribution comes from integers with $k \sim \frac{1}{2} \log \log X$ prime factors.

Next, ignoring ambiguous classes, these k primes yield 2^k classes (with possible repetition) in \mathcal{C} that represent the given integer n . Hence, roughly speaking, one would expect that typically n is represented by each class of its genus provided $2^k \gg h$, which amounts to

$$h < (\log X)^{\frac{\log 2}{2} - \varepsilon} \quad (0.4)$$

corresponding to alternative (i).

On the other hand, if D is sufficiently large, the 2^k classes will be typically distinct. Assuming some mild form of equidistribution in the class group when varying n , we expect for the number of integers $n < X$ with k prime factors represented by a given class to be of order

$$\frac{2^k}{h} \cdot (0.3) = \frac{X}{h \log X} \frac{(\log \log X)^{k-1}}{(k-1)!} \quad (0.5)$$

with total contribution $O\left(\frac{X}{h}\right)$, attained when $k \sim \log \log X$ (at this level of the discussion, there is no difference between h and \sqrt{D}).

In this paper, we consider only the lower range (i). Our aim is to substantiate further the heuristic discussed above according to which, typically, all classes of the genus of $n \sim X$, n representable by a form of discriminant D , do actually represent n .

More precisely, we prove the following (as consequence of Theorem 2 in [B-G]).

Theorem 2'. *Let D be a negative discriminant satisfying*

$$|D| < (\log X)^{\log 2 - \delta} \quad (0.6)$$

for some fixed $\delta > 0$. Then there is $\delta' = \delta'(\delta) > 0$ such that

$\#\{n \sim X; n \text{ square free, representable by some form of discriminant } D \text{ but not by all forms of the genus}\}$

$$< \frac{X}{(\log X)^{\frac{1}{2} + \delta'}}. \quad (0.7)$$

Note that though [Bl], [B-G] establish (0.1) (in fact in a more precise form, cf. Theorem 5 in [B-G]), their results do not directly pertain to the phenomenon expressed in Theorem 2'. As pointed out in [B-G], it was shown on the other hand by Bernays that almost all integers represented by some form in a given genus can be represented by all forms in the genus, but assuming the much stronger restriction

$$D \ll (\log \log X)^{\frac{1}{2} - \varepsilon}. \quad (0.8)$$

A result in the same spirit was also obtained by Golubeva [Go].

The proof of Theorem 2' rests on a general result from arithmetic combinatorics (Theorem 1 below) that we describe next. Assume G a finite abelian group ($G = \mathcal{C}^2$ in our application) in which group operation will be denoted additively. Given a subset $A \subset G$, we introduce the set

$$s(A) = \left\{ \sum x_i; \{x_i\} \text{ are distinct elements of } A \right\}. \quad (0.9)$$

The issue is then to understand what it means for A that $s(A) \neq G$. It turns out that there are basically two possibilities. In the first, A is contained, up to a bounded number of elements, in a proper subgroup H of G of bounded index $[G:H]$. The second scenario is as follows. There are k elements $x_1, \dots, x_k \in A$ with

$$k < (1 + \varepsilon) \frac{\log |G|}{\log 2} \quad (0.10)$$

and a subset $\Omega_{x_1, \dots, x_k} \subset G$ (determined by x_1, \dots, x_k), such that $A \subset \Omega_{x_1, \dots, x_k}$ and

$$|\Omega_{x_1, \dots, x_k}| < \varepsilon |G| \quad (0.11)$$

(we are assuming here that $|G|$ is large).

To prove Theorem 1, one applies the greedy algorithm. Thus given $x_1, \dots, x_k \in A$, we select $x_{k+1} \in A$ as to optimize the size of $s(x_1, \dots, x_{k+1})$. If we do not reach $s(x_1, \dots, x_k) = G$ with k satisfying (0.10), then

$$A \subset \{x_1, \dots, x_k\} \cup \Omega \quad (0.12)$$

where the elements $x \in \Omega$ have the property that

$$|s(x_1, \dots, x_k, x)| \approx |s(x_1, \dots, x_k)|. \quad (0.13)$$

Assuming Ω fails (0.11), the first alternative is shown to occur. The argument involves combinatorial results, such as a version of the Balog-Szemerédi-Gowers theorem and also Kneser's theorem. The reader is referred to the book [T-V] for background material on the matter.

Once Theorem 1 is established, deriving Theorem 2 is essentially routine. We make use, of course, of Landau's result [L2] (established in [Bl] with uniformity in the discriminant), on the distribution of the primes represented by a given class $C \in \mathcal{C}$ – namely, for \mathcal{P}_C the set of primes represented by a class C ,

$$|\{p \in \mathcal{P}_C; p \leq \xi\}| = \frac{1}{\varepsilon(C)h} \int_1^\xi \frac{dt}{\log t} + C(\xi e^{-c\sqrt{\log \xi}}) \quad (0.14)$$

for $\xi \rightarrow \infty$, with $\varepsilon(C) = 2$ if C is ambiguous and $\varepsilon(C) = 1$ otherwise.

The nontrivial upper bound (0.7) is then obtained by excluding certain additional prime divisors, i.e. satisfying $(\frac{D}{p}) \neq -1$, using standard upper bound sieving.

The same approach permits to obtain a similar result considering now shifted primes, i.e. integers n of the form $n = a + q$ with a fixed and q a prime number. Thus

Theorem 3'. *Under the assumption (0.6), fixing $a \in \mathbb{Z}$, we have*

$|\{q + a \sim X; q \text{ prime, } q + a \text{ squarefree representable by some form of discriminant } D \text{ but not by all forms of the genus}\}|$

$$< \frac{X}{(\log X)^{\frac{3}{2} + \delta'}}. \quad (0.15)$$

On the technical side, only crude sieving bounds are needed for our purpose and they can be obtained by the simple inclusion/exclusion principle without the need of Brun's

theory. The arguments covering the specific problem at hand were included in the paper (see Lemmas 4 and 5), which turned out to be more convenient than searching for a reference. Note that the proof of Lemma 5 involves sieving in the ideals and the required remainder estimates are provided by Landau's extension of the Polya-Vinogradov inequality for Hecke characters [L1].

The motivation behind Theorem 3' lies in a result due to H. Iwaniec [Iw] on the number of shifted primes that are representable by the genus of a quadratic form. This in turn is applicable to counting primes which appear as curvatures in a primitive integer Apollonian circle packing using a method similar to that in [BF], where the authors prove that the integers appearing as curvatures in a primitive integer Apollonian packing make up a positive fraction of \mathbb{Z} .

Specifically, let P be a primitive integer Apollonian packing, and let $a \neq 0$ denote a curvature of a circle in P . From [BF], we have that the set S_a of integers less than X represented by certain shifted binary quadratic forms $f_a(x, y) - a$, where the discriminant $D(f_a) = -4a^2$, correspond to curvatures of circles in P . Let $\mathfrak{P}_a \subset S_a$ denote the set of primes in S_a . We may then compute a lower bound for the number of primes less than X appearing as curvatures in P by bounding

$$\left| \bigcup_a \mathfrak{P}_a \right|$$

where the a 's range over a set of our choice. The aim is to use the $\frac{1}{2}$ -dimensional sieve of Iwaniec to first determine the cardinality of \mathfrak{P}_a . In [Iw], Iwaniec proves upper and lower bounds for the number of primes less than N represented by $\phi(x, y) + A$, where $\phi(x, y)$ is a positive definite binary quadratic form and A is an integer. He shows

$$\frac{X}{(\log X)^{3/2}} \ll S(X, \phi, a) \ll \frac{X}{(\log X)^{3/2}}$$

where $S(X, \phi, a)$ denotes the number of primes less than X represented by $\phi(x, y) + A$. Here the discriminant of ϕ is fixed, and the bounds above are obtained by considering the count over all forms in the genus of ϕ : namely, for fixed discriminant, bounds for $S(X, \phi, a)$ are easily derived from bounds for

$$S_1(X, \phi, a) = \sum_{\substack{p \leq X \\ (x, y) = 1, f \in R_\phi \\ p = f(x, y) + a}} 1$$

where R_ϕ denotes the genus of ϕ . In order to apply this to finding bounds for $|\mathfrak{P}_a|$ where a is allowed to grow with X , we must understand both how $S_1(X, \phi, a)$ depends on the discriminant of ϕ , and how S relates to S_1 in the case that D is not fixed. The latter is explained by Theorem 3' for D satisfying (0.6), while the former is done via a careful analysis of the dependence on the discriminant in [Iw] for $D < \log X$. This is discussed in the Appendix. Note that in the application to Apollonian packings, the discriminant of ϕ is always of the form $-4a^2$, but our results apply to a much more general discriminant.

Indeed, Theorem 1 in [Iw] combined with Theorem 3' above implies the following

Corollary 4. *Let $D < 0$ satisfy (0.6) and f be a primitive positive definite binary form of discriminant D . Then*

$$|\{q + a \sim X; q \text{ prime, } q + a \text{ representable by } f\}| \gg \frac{X}{(\log X)^{\frac{3}{2} + \varepsilon}} \quad (0.16)$$

(we assume here $a \in \mathbb{Z}$ fixed for simplicity).

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§1. A result in combinatorial group theory

The aim of this section is to prove Theorem 1 below.

(1). Let G be an abelian group, $|G| = h'$.

For $A \subset G$, denote

$$s(A) = \left\{ \sum x_i; \{x_i\} \text{ distinct elements of } A \right\} \quad (1.0)$$

(the set of sums of distinct elements of A .)

Assume $s(A) \neq G$. We would like to specify the structure of such A .

Start with the following algorithm.

Take $x_1 \in A$.

Assume obtained x_1, \dots, x_j , take x_{j+1} as to maximize

$$s(x_1, \dots, x_{j+1}).$$

Let $\delta_j = \frac{|s(x_1, \dots, x_j)|}{h'}$.

One has

$$\begin{aligned} |s(x_1, \dots, x_j, x)| &= |s(x_1, \dots, x_j) \cup (s(x_1, \dots, x_j) + x)| \\ &= 2|s(x_1, \dots, x_j)| - |s(x_1, \dots, x_j) \cap (s(x_1, \dots, x_j) + x)|. \end{aligned}$$

Hence

$$\mathbb{E}_x[|s(x_1, \dots, x_j, x)|] = 2\delta_j h' - \delta_j^2 h' = \delta_j(2 - \delta_j)h'. \quad (1.1)$$

On the other hand, for all x

$$|s(x_1, \dots, x_j, x)| \leq (2\delta_j)h'. \quad (1.2)$$

Fix $\varepsilon > 0$.

For $\delta_j < \frac{1}{2}$, define

$$\Omega = \{x \in G; |s(x_1, \dots, x_j, x)| < (2 - \varepsilon)\delta_j h'\}.$$

Then, from (1.1), (1.2)

$$\mathbb{E}_x[|s(x_1, \dots, x_j, x)|] \leq (2\delta_j h') \left(1 - \frac{|\Omega|}{h'}\right) + (2 - \varepsilon)\delta_j h' \frac{|\Omega|}{h'}$$

implying

$$|\Omega| < \frac{\delta_j}{\varepsilon} h'. \quad (1.3)$$

For $\delta_j > \frac{1}{2}$, define

$$\Omega = \{x \in G; |s(x_1, \dots, x_j, x)| < (1 - (1 - \delta_j)^{3/2})h'\}.$$

Similarly

$$|\Omega| < (1 - \delta_j)^{1/2}h'. \quad (1.4)$$

It follows from (1.3) that either

(1.5) there exist $x_1, \dots, x_k \in A$ s.t.

$$|s(x_1, \dots, x_k)| > \varepsilon^2 h' \quad (1.6)$$

with

$$k < \frac{\log h'}{\log 2 - \frac{\varepsilon}{2}} \quad (1.7)$$

or

(1.8) There exists elements $x_1, \dots, x_k \in A$ and a set $\Omega_{x_1, \dots, x_k} \subset G$ satisfying

$$A \subset \{x_1, \dots, x_k\} \cup \Omega_{x_1, \dots, x_k} \quad (1.9)$$

$$k < \frac{\log h'}{\log 2 - \frac{\varepsilon}{2}} \quad (1.10)$$

$$|\Omega_{x_1, \dots, x_k}| < \varepsilon h'. \quad (1.11)$$

(2). Let $A_1 \subset A$ s.t.

$$\delta h' = s(A_1) > \varepsilon^2 h'. \quad (2.1)$$

Fix $\varepsilon_1 > 0$ and define

$$\Omega = \{x \in G; |s(A_1 \cup \{x\})| < (1 - \varepsilon_1)|s(A_1)| + \varepsilon_1 h'\}. \quad (2.2)$$

If $(A \setminus A_1) \cap \Omega^c \neq \emptyset$, we add an element and increase the density from δ in (2.1) to $(1 - \varepsilon_1)\delta + \varepsilon_1$.

Assume this process can be iterated r times.

We obtain a set A'_1 such that $s(A'_1)$ has density at least δ' satisfying

$$1 - \delta' = (1 - \varepsilon_1)^r(1 - \delta)$$

and thus $|s(A'_1)| > (1 - \varepsilon^2)h'$ for

$$r \sim \frac{\log \frac{1}{\varepsilon}}{\varepsilon_1}. \quad (2.3)$$

Continuing the process with A'_1 and $\delta > 1 - \varepsilon^2 > \frac{1}{2}$ gives a subset $A''_1 \subset A$ so that $s(A''_1) = G$ and

$$|A''_1| \leq \frac{\log h'}{(\log 2) - \varepsilon} + c \frac{\log \frac{1}{\varepsilon}}{\varepsilon_1} + \log \log h' \quad (2.4)$$

unless we are in alternative (1.8) with (1.10) replaced by (2.4).

Thus it remains to analyze the case when the iteration fails.

If $|\Omega| < \varepsilon h'$, we are again in the situation (1.8) with (1.10) replaced by

$$\frac{\log h'}{\log 2 - \varepsilon} + c \frac{\log \frac{1}{\varepsilon}}{\varepsilon_1}.$$

Assume next Ω defined in (2.2) satisfies

$$|\Omega| > \varepsilon h'. \quad (2.5)$$

Denoting $B = s(A'_1)$, we have by (2.1) and definition of Ω that

$$|B| > \varepsilon^2 h' \quad (2.6)$$

and

$$|B \cap (B + x)| > (1 - \varepsilon_1 \varepsilon^{-2})|B| \text{ for } x \in \Omega. \quad (2.7)$$

Hence

$$1_B * 1_{-B} > (1 - \varepsilon_1 \varepsilon^{-2})|B| \text{ on } \Omega \quad (2.8)$$

implying in particular that

$$|B| > (1 - \varepsilon_1 \varepsilon^{-2})|\Omega|. \quad (2.9)$$

(3). Assume (2.6)-(2.9).

Thus

$$\langle 1_B, 1_B * 1_\Omega \rangle = \langle 1_B * 1_{-B}, 1_\Omega \rangle \geq (1 - \varepsilon_1 \varepsilon^{-2})|B| |\Omega| \quad (3.1)$$

and

$$\|1_B * 1_\Omega\|_2 \geq (1 - \varepsilon_1 \varepsilon^{-2})|B|^{\frac{1}{2}}|\Omega|.$$

Squaring and using the fact that Ω is symmetric

$$\|1_B * 1_\Omega * 1_\Omega\|_2 \geq (1 - \varepsilon_1 \varepsilon^{-2})^2 |B|^{\frac{1}{2}} |\Omega|^2$$

and for any given r (= power of 2)

$$\|1_B * 1_\Omega^{(r)}\|_2 \geq (1 - \varepsilon_1 \varepsilon^{-2})^r |B|^{\frac{1}{2}} |\Omega|^r. \quad (3.2)$$

(where $1_\Omega^{(r)}$ denotes the r fold convolution).

We will rely on the following

Lemma 1. *Let μ be a probability measure on a discrete additive group G and assume (for small κ)*

$$\|\mu * \mu\|_2 > (1 - \kappa)\|\mu\|_2. \quad (3.3)$$

Then there is a subgroup H of G s.t.

$$\frac{1}{2}\|\mu\|_2^{-2} < |H| < 2\|\mu\|_2^{-2} \quad (3.3')$$

and for some $z \in G$

$$\left\| \mu - \frac{1_{H-z}}{|H|} \right\|_1 < c\kappa^{1/12}. \quad (3.3'')$$

Proof.

From (3.3) we have

$$\sum_x \left| \sum_y \mu(x-y)\mu(y) \right|^2 > (1-\kappa)^2 \|\mu\|_2^2$$

and

$$\sum_{y_1, y_2} \langle \mu_{y_1}, \mu_{y_2} \rangle \mu(y_1)\mu(y_2) > (1-\kappa)^2 \|\mu\|_2^2$$

implying

$$\begin{aligned} \sum \|\mu_{y_1} - \mu_{y_2}\|_2^2 \mu(y_1)\mu(y_2) &< 2(1 - (1-\kappa)^2) \|\mu\|_2^2 \\ &< 4\kappa \|\mu\|_2^2. \end{aligned}$$

Hence there is $y_0 \in G$ s.t.

$$\sum \|\mu_y - \mu_{y_0}\|_2^2 \mu(y) < 4\kappa \|\mu\|_2^2$$

and by translation of μ , we may assume $y_0 = 0$, thus

$$\sum \|\mu_y - \mu\|_2^2 \mu(y) < 4\kappa \|\mu\|_2^2.$$

Denote

$$U = \{y \in G; \|\mu - \mu_y\|_2 < \kappa^{1/3} \|\mu\|_2\}.$$

Hence, from the preceding

$$\mu(G \setminus U) < 4\kappa^{1/3}.$$

Since

$$\frac{1}{|U|} \sum_{y \in U} \|\mu - \mu_y\|_2 < 4\kappa^{1/3} \|\mu\|_2$$

it follows by convexity that

$$\left\| \mu - \mu * \frac{1_U}{|U|} \right\|_2 < 4\kappa^{1/3} \|\mu\|_2$$

and in particular

$$\begin{aligned} \|\mu\|_2 &\leq \frac{1}{|U|^{1/2}} + 4\kappa^{1/3} \|\mu\|_2 \\ \|\mu\|_2 &< \frac{1 + 4\kappa^{1/3}}{|U|^{1/2}}. \end{aligned}$$

Next, write

$$\begin{aligned} \left\| \mu - \frac{1_U}{|U|} \right\|_2^2 &= \|\mu\|_2^2 + \frac{1}{|U|} - 2 \frac{\mu(U)}{|U|} \\ &\leq \frac{2 + 10\kappa^{1/3} - 2(1 - 4\kappa^{1/3})}{|U|} \\ &< \frac{18\kappa^{1/3}}{|U|}. \end{aligned}$$

Hence

$$\left\| \mu - \frac{1_U}{|U|} \right\|_2 < \frac{5\kappa^{1/6}}{|U|^{1/2}} \quad (3.4)$$

and also

$$\begin{aligned} \left\| \mu - \frac{1_U}{|U|} \right\|_1 &\leq \mu(U^c) + \sum_{x \in U} \left| \mu(x) - \frac{1}{|U|} \right| \\ &\leq 4\kappa^{1/3} + |U|^{1/2} \left\| \mu - \frac{1_U}{|U|} \right\|_2 \\ &< 6\kappa^{1/6}. \end{aligned} \quad (3.5)$$

From (3.3), (3.4), (3.5), we have

$$\left\| \frac{1_U}{|U|} * \frac{1_U}{|U|} \right\|_2 > (1 - 20\kappa^{1/6}) \frac{1}{|U|^{1/2}}$$

hence

$$E_+(U, U) = \|1_U * 1_U\|_2^2 > (1 - 40\kappa^{1/6}) \cdot |U|^3$$

where E_+ refers to the additive energy.

We apply now some results from arithmetic combinatorics.

First, by (2.5.4), p.82 from [TV] (B-S-G in near-extreme case), there are subsets $U', U'' \subset U$ s.t.

$$|U'|, |U''| > (1 - 10\kappa^{1/12})|U|$$

and

$$|U' - U''| < (1 + 20\kappa^{1/12})|U|.$$

Thus from Ruzsa's triangle inequality, also

$$\begin{aligned} |U' - U'| &\leq \frac{|U' - U''|^2}{|U''|} < (1 + 60\kappa^{1/12})|U| \\ &< (1 + 80\kappa^{1/12})|U|. \end{aligned} \quad (3.6)$$

Next, we apply Kneser's theorem (see [TV], Theorem 5.5, p. 200).

For $T \subset G$, denote

$$\text{Sym}_1(T) = \{x \in G; T + x = T\}$$

the symmetry group of T .

Then by Kneser's theorem, see [T-V]

$$|T - T| \geq 2|T| - |\text{Sym}_1(T - T)|$$

and application with $T = U'$ gives

$$|\text{Sym}_1(U' - U')| > (1 - 80\kappa^{1/12})|U'|. \quad (3.7)$$

Denoting $H = \text{Sym}_1(U' - U)$, $H \subset U' - U$ and thus

$$\begin{aligned} |H| |U'| &\leq \sum_{z \in U' - U' - U'} |H \cap (U' + z)| \\ &\leq |U' - U' - U'| \max_z |H \cap (U' + z)| \\ &< (1 + 300\kappa^{1/12}) |U'| \max_z |H \cap (U' + z)| \end{aligned}$$

from (3.6) and sumset inequalities. Therefore, there is some $z \in G$ s.t.

$$|(H - z) \cap U'| > (1 + 300\kappa^{1/12})^{-1} |H|$$

and in view of (3.7)

$$|U' \Delta (H - z)| < 1000\kappa^{\frac{1}{12}} |U|$$

and

$$|U \Delta (H - z)| < 1000\kappa^{\frac{1}{12}} |U|. \quad (3.8)$$

From (3.5), (3.8) we have

$$\left\| \mu - \frac{1_{H-z}}{|H|} \right\|_1 < C\kappa^{\frac{1}{12}}. \quad (3.9)$$

From (3.4), (3.8), we obtain (3.31) proving Lemma 1.

Returning to (2.5), (3.2), we have that

$$\left\| \left(\frac{1_\Omega}{|\Omega|} \right)^{(r)} \right\|_2$$

decreases in r and is between $\frac{1}{\sqrt{h}}$ and $\frac{1}{\sqrt{\varepsilon h'}}$. Hence there is some τ

$$\log r < \frac{c}{\kappa} \log \frac{1}{\varepsilon} \quad (3.10)$$

such that $\mu = \left(\frac{1_\Omega}{|\Omega|} \right)^{(r)}$ satisfies (3.3).

From (3.2), (3.3'), we conclude that

$$\begin{aligned} \left\| 1_B * \frac{1_H}{|H|} \right\|_2 &\geq ((1 - \varepsilon_1 \varepsilon^{-2})^r - c\kappa^{1/12}) |B|^{1/2} \\ &> (1 - c\kappa^{1/12}) |B|^{1/2} \end{aligned} \quad (3.11)$$

provided

$$\varepsilon_1 < \left(\frac{1}{\varepsilon} \right)^{c\kappa^{-1}}. \quad (3.12)$$

Also, from (3.3') and the preceding

$$|H| > \frac{1}{2} |\Omega| > \frac{\varepsilon}{2} h'. \quad (3.13)$$

Let $\{H_\alpha\}$ be the cosets of $H \subset G$. Then

$$\|1_B * 1_H\|_2^2 = \sum_{\alpha} \|1_{(B \cap H_\alpha)} * 1_H\|_2^2.$$

Let $\kappa_1 > 0$ be a small parameter and define

$$I_0 = \{\alpha; |B \cap H_\alpha| > (1 - \kappa_1)|H|\}$$

and I_1 the complement.

One has

$$\|1_{(B \cap H_\alpha)} * 1_H\|_2^2 = E_+(H, B \cap H_\alpha) \leq |B \cap H_\alpha|^2 \cdot |H|$$

and hence, by (3.11)

$$\begin{aligned} (1 - c\kappa^{1/12})|B| \cdot |H|^2 &\leq |H| \sum_{\alpha} |B \cap H_\alpha|^2 \\ &\leq |H| \left(\sum_{\alpha \in I_0} |H| |B \cap H_\alpha| + (1 - \kappa_1) \sum_{\alpha \in I_1} |H| |B \cap H_\alpha| \right) \\ &\leq |H|^2 (|B| - \kappa_1 \sum_{\alpha \in I_1} |B \cap H_\alpha|). \end{aligned}$$

Hence $B = B_0 \cup B_1$ with

$$|B_1| = \sum_{\alpha \in I_1} |B \cap H_\alpha| < c\kappa^{1/12} \kappa_1^{-1} |B|. \quad (3.14)$$

Assume

$$\kappa \ll \kappa_1^{12} \quad (3.15)$$

so that in particular $I_0 \neq \emptyset$.

Let $y \in A \setminus A'_1$. Then $y \in \Omega$ and by (2.7)

$$|B \cap (B + y)| > (1 - \varepsilon_1 \varepsilon^{-2})|B|.$$

Let $\varphi : G \rightarrow G/H = I_0 \cup I_1$.

If $\alpha \in I_0$, then

$$\begin{aligned} |((B \cap H_\alpha) + y) \cap B| &\geq |(B + y) \cap B| - \sum_{\alpha' \neq \alpha} |B \cap H_{\alpha'}| \\ &> (1 - \varepsilon_1 \varepsilon^{-2})|B| - |B| + |B \cap H_\alpha| \\ &> (1 - \kappa_1)|H| - \varepsilon_1 \varepsilon^{-2}|B| \\ &\stackrel{(3.13)}{>} (1 - \kappa_1 - 2\varepsilon_1 \varepsilon^{-3})|H|. \end{aligned}$$

Thus certainly

$$|H_{\alpha+\varphi(y)} \cap B| > (1 - \kappa_1 - 2\varepsilon_1 \varepsilon^{-3})|H|.$$

From (3.14), if $\beta \in I_1$

$$|H_\beta \cap B| < c\kappa^{1/12}\kappa_1^{-1}h \underset{(3.13)}{<} c\kappa^{1/12}\kappa_1^{-1}\varepsilon^{-1}|H|. \quad (3.13)$$

Assume

$$\varepsilon_1 < 10^{-3}\varepsilon^3 \quad (3.16)$$

and

$$\kappa \ll \kappa_1^{24}\varepsilon^{12} \quad (3.17)$$

(\leftrightarrow 3.15).

It follows that $|H_\beta \cap B| < \kappa_1|H|$ for $\beta \notin I_0$ while certainly

$$|H_{\alpha+\varphi(y)} \cap B| > \frac{1}{2}|H|.$$

Hence $\alpha + \varphi(y) \in I_0$ and we proved that

$$I_0 + \varphi(y) = I_0 \text{ in } G/H \text{ for all } y \in A \setminus A'_1.$$

Thus

$$\varphi(A \setminus A'_1) \subset \text{Sym}_1(I_0) \text{ in } G/H. \quad (3.18)$$

We distinguish two cases.

If $I_0 = G/H$, then $|B| = |s(A'_1)| > (1 - \kappa_1)h'$. We may then construct A''_1 as in §2 and conclude (1.8) with $k < (2.4)$, $|\Omega| < \sqrt{\kappa_1}h'$.

Assume next $I_0 \neq G/H$. Hence $\text{Sym}_1(I_0) \neq G/H$ and $H' = \varphi^{-1}(\text{Sym}_1(I_0)) \supset H$ is a proper subgroup of G . Hence

$$\frac{\varepsilon}{2}h' < |H'| \leq \frac{h'}{2}.$$

By (3.18),

$$A \setminus A'_1 \subset H'.$$

Since I_0 is a union of cosets of $\text{Sym}_1(I_0)$ in G/H , $\varphi^{-1}(I_0)$ is a union of cosets H'_τ of H' , each satisfying

$$|B \cap H'_\tau| > (1 - \kappa_1)|H'| \text{ for } \tau \in I'_0$$

(by definition of I_0), where $I_0 = \bigcup_{\tau \in I'_0} \text{Sym}_1(I_0)_\tau$.

Thus we may identify H and H' and write

$$A \setminus A'_1 \subset H$$

with

$$\frac{\varepsilon}{2}h' < |H| < \frac{h'}{2}. \quad (3.19)$$

The set $s(A'_1) = B_0 \cup B_1$ with

$$B_0 = \bigcup_{\alpha \in I_0} (s(A'_1) \cap H_\alpha) \text{ and } B_1 = \bigcup_{\alpha \in I_1} (s(A'_1) \cap H_\alpha)$$

and

$$|s(A'_1) \cap H_\alpha| > (1 - \kappa_1)|H| \text{ for } \alpha \in I_0 \quad (3.20)$$

$$|B_1| < c\kappa^{1/24}h' \quad (3.21)$$

$$I_0 \neq \emptyset, I_0 \neq G/H. \quad (3.22)$$

Next, take a set $z_1, \dots, z_r \in A'_1, r < \frac{2}{\varepsilon}$ of representatives for $\varphi(A'_1)$ and denote $A_2 = A'_1 \setminus \{z_1, \dots, z_r\}$. Then

$$s(A_2) \subset s(A'_1) \text{ and } |s(A_2)| \geq 2^{-r}|s(A'_1)|.$$

Thus there is some $\alpha \in G/H$ s.t.

$$|s(A_2) \cap H_\alpha| > \frac{\varepsilon}{2}|s(A_2)| > \varepsilon 2^{-r-1}|s(A'_1)| > \varepsilon 2^{-r-2}h'.$$

Hence, for each $z \in s(z_1, \dots, z_r)$

$$|s(A'_1) \cap H_{\alpha+\varphi(z)}| \geq |(s(A_2) + z) \cap H_{\alpha+\varphi(z)}| > \varepsilon 2^{-n-2}h'.$$

We claim that $\alpha + \varphi(z) = \beta \in I_0$. Otherwise, $\beta \in I_1$ and $s(A'_1) \cap H_\beta \subset B_1$, implying by (3.21) that

$$|s(A'_1) \cap H_\beta| < c\kappa^{1/24}h'$$

and this is impossible, provided

$$\kappa < 2^{-\frac{100}{\varepsilon}}. \quad (3.23)$$

Hence

$$I_0 \supset \alpha + \varphi(s(z_1, \dots, z_r)) = \alpha + \varphi(s(A'_1))$$

and since $I_0 \subset \varphi(s(A'_1))$, by (3.20), it follows that $I_0 = \varphi(s(A'_1))$ and therefore by (3.22)

$$\varphi(s(A'_1)) \neq G/H. \quad (3.24)$$

Next partition

$$I_0 = \varphi(s(A'_1)) = J \cup J'$$

with

$$J = \left\{ \alpha \in G/H, |A'_1 \cap H_\alpha| > \frac{10}{\varepsilon} \right\}.$$

Thus

$$\left| \bigcup_{\alpha \in J'} (A'_1 \cap H_\alpha) \right| < \frac{20}{\varepsilon^2}. \quad (3.25)$$

Take elements $\mathcal{Z} = \{z_{\alpha,t}; \alpha \in J, t \leq \frac{10}{\varepsilon}\} \cup \{z_\alpha; \alpha \in J'\}$ with $\varphi(z_{\alpha,t}) = \alpha$.

Then

$$s(A'_1) \supset s(\mathcal{Z})$$

and

$$\varphi(s(A'_1)) \supset \left\{ \sum_{\alpha \in J} u_\alpha \alpha; 0 \leq u_\alpha \leq \frac{10}{\varepsilon} \right\} + J' = \langle J \rangle + J'$$

where $\langle J \rangle$ is the group generated by $J \subset G/H$. Thus $|\langle J \rangle| \leq |\varphi(s(A'_1))|$.

From (3.24), $\langle J \rangle \neq G/H$ and $H' = \varphi^{-1}(\langle J \rangle)$ is a proper subgroup of G .

Hence, by (3.25)

$$|A'_1 \setminus H'| < c(\varepsilon) \quad (3.26)$$

and since $A \setminus A'_1 \subset H$,

$$|A \setminus H'| < c(\varepsilon)$$

with H' a proper subgroup of G , $[G : H'] \leq \frac{2}{\varepsilon}$.

Recalling the constraints (3.12), (3.15), (3.16), (3.17), (3.23) on the parameters $\varepsilon, \varepsilon_1, \kappa, \kappa_1$, take

$$\begin{aligned} \kappa_1 &= \varepsilon^2 \\ \kappa &= 2^{-\frac{100}{\varepsilon}} \\ \varepsilon_1 &= \left(\frac{1}{\varepsilon}\right)^{C \cdot 2^{\frac{100}{\varepsilon}}}. \end{aligned}$$

(4). Summarizing the preceding, we proved the following.

Theorem 1.

Let G be a finite abelian group and $A \subset G$, $|G| = h'$. Let $\varepsilon > 0$ be a small constant.

There are the following alternatives.

$$(4.1) \quad s(A) = G$$

$$(4.2) \quad \text{There is a proper subgroup } H \text{ of } G, \text{ such that}$$

$$[G : H] < \frac{2}{\varepsilon} \text{ and } |A \setminus H| < c(\varepsilon).$$

$$(4.3) \quad \text{There are } k \text{ elements } x_1, \dots, x_k \in A \text{ and a subset } \Omega_{x_1, \dots, x_k} \subset G \text{ depending only on } x_1, \dots, x_k, \text{ such that}$$

$$k < (1 + \varepsilon) \frac{\log h'}{\log 2} + c \log \log h' + c(\varepsilon) \quad (4.4)$$

$$|\Omega_{x_1, \dots, x_k}| \leq \varepsilon h' + k \quad (4.5)$$

and

$$A \subset \Omega_{x_1, \dots, x_k}. \quad (4.6)$$

§2. Application to the class group

(5). We apply the preceding to the class group \mathcal{C} for a large discriminant $D < 0$.

Let $n \in \mathbb{Z}_+$ be square free; $n = \prod p_j$ with $(p_j, D) = 1$ and $\mathcal{X}_D(p_j) \neq -1$. Let C_j, C_j^{-1} be the classes that represent p_j . Then n is representable by all classes in the formal expansion $\prod \{C_j, C_j^{-1}\}$ (see [Bl], Cor. 2.3).

Let $G = \mathcal{C}^2$. Thus $h' = |G| = h/g$ with $g = |\mathcal{C}/\mathcal{C}^2|$ the number of genera. Let $A = \{C_j^2\} \subset G$. We have

$$\prod \{C_j; C_j^{-1}\} = \left(\prod C_j^{-1} \right) s(A) \quad (5.1)$$

with $s(A)$ defined as in (1.0).

Fix $\varepsilon > 0$ a small parameter and apply Theorem 1 to $A \subset G$.

If $s(A) = G$ as in (4.1) of Theorem 1, then

$$\prod \{C_j, C_j^{-1}\} = \left(\prod C_j^{-1} \right) \mathcal{C}^2.$$

Since $\mathcal{C}/\mathcal{C}^2$ is the group \mathcal{G} of the genera, it follows that in this case n is representable by any form of the genus if it's representable by some form.

Assume now that A satisfies the conditions of alternative (4.2) of Theorem 1.

Denote $\eta : \mathcal{C} \rightarrow \mathcal{C}^2$ obtained by squaring and let $\mathcal{C}' = \eta^{-1}(H)$. Since \mathcal{C} is a proper subgroup of \mathcal{C}_1 , we have

$$\frac{\varepsilon}{2}h < |\mathcal{C}'| \leq \frac{h}{2}$$

where $h = |\mathcal{C}|$ is the class number.

There is a set of indices \mathcal{J} such that $|\mathcal{J}| < C(\varepsilon)$ and for $j \notin \mathcal{J}$, $C_j^2 \in H$, hence $C_j, C_j^{-1} \in \mathcal{C}'$.

Denote \mathcal{P}_C the primes represented by the class C . Thus $\mathcal{P}_C = \mathcal{P}_{C^{-1}}$.

It follows from the preceding that $n(\prod_{j \in \mathcal{J}} p_j)^{-1}$ has all its prime factors in the set

$$\mathcal{P}(\mathcal{C}') \equiv \bigcup_{C \in \mathcal{C}'} \mathcal{P}_C.$$

We recall the following distributional theorem.

Lemma 2. (Landau; [Bl], Lemma 5.1).

Assume $D < (\log \xi)^A$, A fixed.

Then

$$|\{p \in \mathcal{P}_C; p \leq \xi\}| = \pi_C(\xi) = \frac{1}{\varepsilon(C)h} \int_1^\xi \frac{dt}{\log t} + O(\xi e^{-c\sqrt{\log \xi}}) \quad (5.2)$$

with $\varepsilon(C) = 2$ if C is ambiguous and $\varepsilon(C) = 1$ otherwise.

Recall also that the number of ambiguous classes equals

$$\gamma = \#(\mathcal{C}/\mathcal{C}^2) = \text{number of genera} \ll 2^{\omega(D)}.$$

Hence from (5.2)

$$\begin{aligned} \pi_{\mathcal{C}'}(\xi) &= |\{p \in \mathcal{P}(\mathcal{C}'); p \leq \xi\}| \\ &\leq \sum_{C \text{ ambiguous}} \pi_C(\xi) + \frac{1}{2} \sum_{\substack{C \in \mathcal{C}' \\ \text{not ambiguous}}} \pi_C(\xi) \\ &\leq (\gamma + |\mathcal{C}'|) \frac{1}{2h} \int_2^\xi \frac{dt}{\log t} + O(\xi e^{-c\sqrt{\log \xi}} h) \end{aligned}$$

and since $|\mathcal{C}'| \leq \frac{h}{2}$ and $h < D^{\frac{1}{2}+\varepsilon} < (\log \xi)^A$

$$< \left(\frac{1}{4} + \frac{1}{h^{1-\varepsilon}} \right) \int_2^\xi \frac{dt}{\log t}. \quad (5.3)$$

Thus, in summary, the number of integers $n \leq X$ obtained in alternative (4.2), is at most

$$\sum_{\substack{r \leq C_\varepsilon; p_1 \dots p_r < X \\ \mathcal{C}' < \mathcal{C} \\ 2 \leq |\mathcal{C}; \mathcal{C}'| \leq \frac{2}{\varepsilon}}} \# \left\{ n \leq \frac{X}{p_1 \dots p_r}; n \text{ square free with prime factors in } \mathcal{P}(\mathcal{C}') \right\} \quad (5.4)$$

with $\mathcal{P}(\mathcal{C}')$ satisfying (5.3) and $\{p_1, \dots, p_r\}$ unordered and distinct, with $\mathcal{X}_D(p_j) \neq -1$.

To bound the expressions $\#\{\dots\}$, use the upper bound sieve.

For instance Corollary 6.2 in [I-K], which we apply with $\mathcal{A} = \mathbb{Z}_+$ and considering

$$P(z) = \prod_{\substack{p \notin \mathcal{P}(\mathcal{C}') \\ p < z}} p. \quad (5.5)$$

Hence $g(d) = \frac{1}{d}$, $|r_d(\mathcal{A})| \leq 1$, $\kappa = 1$, $K = 1$,

$$V(z) = \prod_{p|p(z)} (1 - g(p)) = \prod_{\substack{p < z \\ p \notin \mathcal{P}(\mathcal{C}')}} \left(1 - \frac{1}{p} \right) \quad (5.6)$$

and from [IK], (6.2), (6.80), applied with $D = z$, $s = 1$

$$\#\{n < X; (n, p(z)) = 1\} < CXV(z) + R(z) \quad (5.7)$$

with

$$R(z) = \sum_{d|P(z), d < z} |r_d(\mathcal{A})| \leq z. \quad (5.8)$$

Using (5.3) and partial summation

$$\begin{aligned} \sum_{\substack{p < z \\ p \notin \mathcal{P}(\mathcal{C}')}} \frac{1}{p} &> \sum_{u < z} \frac{1}{u^2} |\{p \leq u; p \notin \mathcal{P}(\mathcal{C}')\}| = \sum_{u < z} \frac{1}{u^2} \left(\frac{u}{\log u} - \pi_{\mathcal{C}'}(u) \right) + O(1) \\ &> \sum_{\exp(h^{1/A}) < u < z} \left(\frac{3}{4} - \frac{1}{h^{1-\varepsilon}} \right) \frac{1}{u \log u} + O(1) \\ &> \left(\frac{3}{4} - \frac{1}{h^{1-\varepsilon}} \right) \log \left(\frac{\log z}{h^{1/A}} \right). \end{aligned} \quad (5.9)$$

Hence

$$V(z) \lesssim \exp \left(- \left(\sum_{\substack{p < z \\ p \notin \mathcal{P}(\mathcal{C}')}} \frac{1}{p} \right) \right) < \left(\frac{h^{1/A}}{\log z} \right)^{\frac{3}{4} - o(1)} \quad (5.10)$$

for $z > \exp(D^{1/A})$.

Substituting in (5.7) with $z = \sqrt{Y}$ gives for $Y > \exp D^{1/A}$

$$\#\{n < Y, n \text{ squarefree with prime factors in } \mathcal{P}(\mathcal{C}')\} \lesssim \frac{h^{1/A}}{(\log Y)^{3/4-o(1)}} Y \quad (5.11)$$

(here A is an arbitrary large fixed constant).

Returning to (5.4), we have for $\tau > 0$ fixed, $X^\tau > \exp(D^{1/A})$

$$\begin{aligned} & \sum_{\substack{p_1 \dots p_r < X^{1-\tau} \\ \mathcal{X}_D(p_j) \neq 1}} \#\left\{n \leq \frac{X}{p_1 \dots p_r}; n \text{ square free with primes in } \mathcal{P}(\mathcal{C}')\right\} \\ & \stackrel{(5.11)}{\lesssim} \frac{h^{1/A} X}{\tau(\log X)^{3/4-o(1)}} \sum_{\substack{p_1 \dots p_r < X \\ \mathcal{X}_D(p_j) \neq -1}} \frac{1}{p_1 \dots p_r} \\ & \lesssim \frac{h^{1/A} X}{\tau(\log X)^{3/4-o(1)}} \frac{(\frac{1}{2} \log \log X)^r}{r!} \\ & \lesssim \frac{h^{1/A} X}{\tau(\log X)^{\frac{3}{4}-o(1)}} \end{aligned} \quad (5.12)$$

since $r < C(\varepsilon)$.

This gives the contribution

$$\ll \#\left\{\mathcal{C}' < \mathcal{C}; [\mathcal{C} : \mathcal{C}'] \leq \frac{2}{\varepsilon}\right\} \cdot \frac{h^{o(1)} X}{\tau(\log X)^{\frac{3}{4}-o(1)}}. \quad (5.13)$$

It remains to consider the case (*): $n < X$ with prime divisors p_1, \dots, p_r such that $p_1 \dots p_r > X^{1-\tau}$.

Lemma 3. *Fix $r \in \mathbb{Z}_+$. Then, for X large enough*

$$\begin{aligned} & |\{n < X; n \text{ represented by } \mathcal{C} \text{ and product of at most } r \text{ distinct primes}\}| \\ & < \frac{rX}{\log X} \left(\frac{e(\frac{1}{2} + \varepsilon) \log \log X}{r-1} \right)^{r-1}. \end{aligned} \quad (5.14)$$

Proof.

We get the estimate

$$\begin{aligned} & \sum_{\substack{p_1 < \dots < p_{r-1} \\ \mathcal{X}_D(p_j) \neq -1}} \frac{X(p_1 \dots p_{r-1})^{-1}}{\log(X(p_1 \dots p_{r-1})^{-1})} \\ & < \frac{rX}{\log X} \sum_{\substack{p_1 < \dots < p_{r-1} < X \\ \mathcal{X}_D(p_j) \neq -1}} \frac{1}{p_1 \dots p_{r-1}} < \\ & \frac{rX}{(r-1)! \log X} \left(\sum_{\substack{p < X \\ \mathcal{X}_D(p) \neq -1}} \frac{1}{p} \right)^{r-1}. \end{aligned} \quad (5.15)$$

From Lemma 2 and partial summation

$$\begin{aligned}
& \sum_{\substack{p < X \\ \mathcal{X}_D(p) \neq -1}} \frac{1}{p} = \frac{1}{2} \sum_{C \text{ non-ambiguous}} \sum_{\substack{p \in \mathcal{P}_C \\ p < X}} \frac{1}{p} + \sum_{C \text{ ambiguous}} \sum_{\substack{p \in \mathcal{P}_C \\ p < X}} \frac{1}{p} \\
& \leq (h - |\mathcal{G}|) \left[\frac{1}{2h} \int_2^X \frac{1}{y^2} \left(\int_2^y \frac{dt}{\log t} \right) dy + c_A \int_{\exp(D^{1/A})}^X \frac{1}{y} e^{-c\sqrt{\log y}} dy \right] \\
& + |\mathcal{G}| \left[\frac{1}{2h} \int_2^X \frac{1}{y^2} \left(\int_2^y \frac{dt}{\log t} \right) dy + C_A \int_{\exp(D^{1/A})}^X \frac{1}{y} e^{-\sqrt{\log y}} dy \right] \\
& + \int^{\exp(D^{1/A})} \frac{1}{y^2} \frac{y}{\log y} dy \\
& < \frac{1}{2} \int_2^X \frac{1}{y^2} \left(\frac{y}{\log y} + O\left(\frac{y}{(\log y)^2}\right) \right) dy + C_A h e^{-c|D|^{(A/2)}} + \frac{\log |D|}{A} \\
& < \frac{1}{2} \log \log X + \frac{\log |D|}{A} + O(1) \\
& < \left(\frac{1}{2} + o(1) \right) \log \log X
\end{aligned} \tag{5.16}$$

for X large enough.

Substitution of (5.16) in (5.15) gives by Stirling

$$\frac{rX}{\log X} \left(\frac{e(\frac{1}{2} + \varepsilon) \log \log X}{r - 1} \right)^{r-1}$$

proving Lemma 3.

Returning to the case (*), we obtain the bound

$$\begin{aligned}
& \sum_{\substack{u < X^\tau \\ \text{sq-free represented by } \mathcal{C}}} \frac{rXu^{-1}}{\log X} \left(\frac{e(\frac{1}{2} + \varepsilon) \log \log X}{r - 1} \right)^{r-1} \\
& \sim \frac{rX}{\log X} \left(\frac{e(\frac{1}{2} + \varepsilon) \log \log X}{r - 1} \right)^{r-1} \left\{ \sum_{\substack{u < X^\tau \\ \text{sq represented by } \mathcal{C}}} \frac{1}{u} \right\} \\
& \lesssim \frac{rX}{\log X} \left(\frac{e(\frac{1}{2} + \varepsilon) \log \log X}{r - 1} \right)^{r-1} \left(\int^{X^\tau} \frac{h^{1/A}}{u(\log u)^{\frac{1}{2}-o(1)}} + D^{1/A} \right) \\
& \lesssim rX \left(\frac{e(\frac{1}{2} + \varepsilon) \log \log X}{r - 1} \right)^{r-1} \left(\frac{\sqrt{\tau}}{(\log X)^{\frac{1}{2}-\frac{1}{A}}} + \frac{1}{(\log X)^{1-\frac{1}{A}}} \right).
\end{aligned} \tag{5.17}$$

Recall that $r < C_\varepsilon$. Taking $\tau = (\log X)^{-\frac{1}{\varepsilon}}$, we obtain

$$(5.13) + (5.17) < \# \left\{ \mathcal{C}' < \mathcal{C}; [\mathcal{C} : \mathcal{C}'] \leq \frac{2}{\varepsilon} \right\} \frac{X}{(\log X)^{\frac{1}{2} + \frac{1}{32}}} \tag{5.18}$$

with ε fixed, $D < (\log X)^C$ and X large enough.

Next, consider the contribution from alternative (4.3) of Theorem 1.

This contribution is clearly bounded by

$$\sum_{k \text{ as in (4.4)}} \sum_{\substack{p_1 < \dots < p_k \\ \mathcal{X}_D(p_j) \neq -1 \\ p_1 \dots p_k < X}} \left| \left\{ n < \frac{X}{p_1 \dots p_k}; n \text{ square free with primes in } \mathcal{P}(\eta^{-1}(\Omega_{p_1, \dots, p_k})) \right\} \right| \quad (5.19)$$

where $\Omega_{p_1, \dots, p_k} \subset \mathcal{C}^2$ satisfies by (4.5)

$$|\Omega_{p_1 \dots p_k}| < 2\varepsilon |\mathcal{C}^2|$$

and hence $\tilde{\Omega}_{p_1 \dots p_k} = \eta^{-1}(\Omega_{p_1, \dots, p_k})$ satisfies

$$|\tilde{\Omega}_{p_1 \dots p_k}| < 2\varepsilon h. \quad (5.20)$$

Repeating (5.3)-(5.11) with \mathcal{C}' replaced by $\tilde{\Omega}_{p_1, \dots, p_k}$, we obtain that for $y > \exp(D^{1/A})$

$$\begin{aligned} & |\{n < Y, n \text{ square free with prime factors in } \mathcal{P}(\tilde{\Omega}_{p_1 \dots p_k})\}| \\ & \lesssim \frac{h^{1/A}}{(\log Y)^{1-3\varepsilon}} Y. \end{aligned} \quad (5.21)$$

We will consider several cases.

Assume in (5.19), $p_1 \dots p_k < \sqrt{X}$.

By (5.21), we obtain the bound

$$\begin{aligned} & \frac{h^{1/A} X}{(\log X)^{1-3\varepsilon}} \sum_{\substack{p_1 < \dots < p_k < X \\ \mathcal{X}_D(p_j) = -1}} \frac{1}{p_1 \dots p_k} \\ & < \frac{h^{1/A} X}{(\log X)^{1-3\varepsilon}} \left(\frac{\frac{\varepsilon}{2} \log \log X}{k} \right)^k. \end{aligned} \quad (5.22)$$

By (4.4), $k < (1+2\varepsilon) \frac{\log h}{\log 2}$. At this point, the size of h becomes essential. Write $h = (\log X)^\rho$ and $k = \sigma \log \log X$ with $\sigma < \frac{(1+2\varepsilon)}{\log 2} \rho$.

Then (5.22) becomes

$$\frac{h^{1/A} X}{(\log X)^{1-3\varepsilon}} \left(\frac{e}{2\sigma} \right)^{\sigma \log \log X} = \frac{h^{1/A} X}{(\log X)^{1-3\varepsilon}} (\log X)^{1-\log 2 \sigma} \sigma. \quad (5.23)$$

Assume $\kappa > 4\varepsilon$ and

$$\rho < (1-\kappa) \frac{\log 2}{2}. \quad (5.24)$$

Then

$$\sigma < \left(1 - \frac{\kappa}{2}\right) \frac{1}{2}$$

and hence

$$\begin{aligned}
(5.23) &< \frac{h^{1/A} X}{(\log X)^{1-3\varepsilon}} (\log X)^{\frac{1}{2}(1-\log(1-\frac{\kappa}{2}))(1-\frac{\kappa}{2})} \\
&< \frac{X}{(\log X)^{1-3\varepsilon-\frac{1}{A}}} (\log X)^{\frac{1}{2}-\frac{1}{16}\kappa^2+0(\kappa^3)} \\
&\leq \frac{X}{(\log X)^{\frac{1}{2}+\frac{\kappa^2}{20}}}
\end{aligned} \tag{5.25}$$

provided

$$\kappa > 10\left(\sqrt{\varepsilon} + \frac{1}{\sqrt{A}}\right). \tag{5.26}$$

Next, assume in (5.19) that $p_1 \dots p_k > \sqrt{X}$. Hence

$$p_k > X^{\frac{1}{2k}}.$$

Rewrite the p_1, \dots, p_k sum in (5.19) as

$$\sum_{\substack{p_1 < \dots < p_{k-1} \\ \mathcal{X}_D(p_j) \neq -1 \\ p_1 \dots p_{k-1} < X^{1-\frac{1}{2k}}}} \sum_{X^{\frac{1}{2k}} < p_k < \frac{X}{p_1 \dots p_{k-1}}} \left| \left\{ n < \frac{X}{p_1 \dots p_k}; n \text{ sf with primes in } \mathcal{P}(\tilde{\Omega}_{p_1 \dots p_k}) \right\} \right|. \tag{5.27}$$

Fix $p_1 \dots p_k$, and denote $X' = \frac{X}{p_1 \dots p_{k-1}} > X^{\frac{1}{2k}}$.

If $\frac{X}{p_1 \dots p_k} \leq \exp |D|^{1/A}$, then $\frac{X'}{\exp D^{1/A}} < p_k < X'$ and we obtain the bound

$$\begin{aligned}
X' \sum_{\frac{X'}{\exp(D^{1/A})} < p_k < X'} \frac{1}{p_k} &< X' \{ \log \log X' - \log(\log X' - D^{1/A}) \} \\
&\lesssim X' \frac{D^{1/A}}{\log X'} \leq \frac{kX}{p_1 \dots p_{k-1}} \frac{|D|^{1/A}}{\log X}.
\end{aligned} \tag{5.28}$$

If $\frac{X}{p_1 \dots p_k} > \exp |D|^{1/A}$, apply (5.21) to get the bound

$$h^{1/A} X' \left(\sum_{X^{\frac{1}{2k}} < p_k < X'} \frac{1}{(\log \frac{X'}{p_k})^{1-3\varepsilon}} \frac{1}{p_k} \right). \tag{5.29}$$

If $2^\ell < \log \frac{X'}{p_k} < 2^{\ell+1}$, then $p_k > X' \cdot e^{-2^\ell}$ and we obtain the contribution

$$\begin{aligned}
&\lesssim \frac{1}{2^{\ell(1-3\varepsilon)}} [\log \log X' - \log \log (X' e^{-2^\ell})] \\
&= -\frac{1}{2^{\ell(1-3\varepsilon)}} \log \left(1 - \frac{2^\ell}{\log X'} \right).
\end{aligned} \tag{5.30}$$

Distinguishing the cases $2^\ell < \frac{1}{2} \log X'$ and $\frac{1}{2} \log X' \leq 2^\ell < \log \frac{X'}{p_k}$ we get

$$(5.30) < \frac{\log \log X}{(\log X')^{1-3\varepsilon}} \lesssim \frac{k \log \log X}{(\log X)^{1-3\varepsilon}} \tag{5.31}$$

and

$$(5.29) < \frac{k|D|^{1/A}(\log \log X)}{(\log X)^{1-3\varepsilon}} \frac{X}{p_1 \dots p_{k-1}} \quad (5.32)$$

that also captures (5.28).

Substitution of (5.32) in (5.27) gives the bound

$$\begin{aligned} & \frac{k|D|^{1/A}(\log \log X)}{(\log X)^{1-3\varepsilon}} X \left(\sum_{\substack{p_1 < \dots < p_{k-1} < X \\ \mathcal{X}_D(p_j) = -1}} \frac{1}{p_1 \dots p_{k-1}} \right) \\ & < \frac{(\log \log X)^2 |D|^{1/A}}{(\log X)^{1-3\varepsilon}} X \left(\frac{\frac{\varepsilon}{2} \log \log X}{k-1} \right)^{k-1} \end{aligned} \quad (5.33)$$

for which the bound (5.25) on (5.22) holds, under the assumption

$$h < (\log X)^{(1-\kappa)\frac{\log 2}{2}} \quad (5.34)$$

with κ satisfying (5.20).

In view of the preceding, in particular estimates (5.18) and (5.25), and taking into account that $|D|^{\frac{1}{2}-\varepsilon} \ll h \ll |D|^{\frac{1}{2}+\varepsilon}$ and the number of genera is bounded by $2^{\omega(D)} \ll |D|^\varepsilon$, we conclude

Theorem 2. *Let $\kappa > 0$ be a fixed constant and $D < 0$ a negative discriminant satisfying*

$$|D| < (\log X)^{(1-\kappa)\log 2}. \quad (5.35)$$

Let \mathcal{C} be the class group. Then for X large enough

$$\begin{aligned} & \#\{n \sim X; n \text{ square free, representable by some form but not by all forms of the genus}\} \\ & \lesssim_\kappa \#\left\{\mathcal{C}' \text{ subgroup of } \mathcal{C}; [\mathcal{C} : \mathcal{C}'] < \frac{10^3}{\kappa^2}\right\} \cdot \frac{X}{(\log X)^{\frac{1}{2} + \frac{1}{33}}} + \frac{X}{(\log X)^{\frac{1}{2} + \frac{\kappa^2}{20}}}. \end{aligned} \quad (5.36)$$

§3. Representation of shifted primes

(6). Next, we establish a version of Theorem 2 for shifted primes.

More precisely we get a bound on

$$\begin{aligned} & \#\{q \sim X \text{ prime; } q+a \text{ square-free and representable by some form but not} \\ & \text{all of the forms of the genus}\} \end{aligned} \quad (6.1)$$

We use a similar strategy, combining the combinatorial Theorem 1 with upper bound sieving. In fact, only the following crude upper bound will be needed.

Lemma 4. *Let $Y \in \mathbb{Z}$ be a large integer and let for each prime $\ell < Y$ a subset $R_\ell \subset \mathbb{Z}/\ell\mathbb{Z}$ be given, $|R_\ell| \in \{0, 1, 2\}$ (for the applications below). Then*

$$\#\{n < Y; \pi_\ell(n) \notin R_\ell \text{ for each } \ell\} < (\log \log Y)^3 \prod_{\ell} \left(1 - \frac{|R_\ell|}{\ell}\right) Y + \frac{Y}{(\log Y)^{10}}. \quad (6.2)$$

Proof.

Denote $\Omega = \{n \in \mathbb{Z}_+; n < Y\}$.

For ℓ prime, let

$$\Omega_\ell = \{n \in \Omega; \pi_\ell(n) \in R_\ell\}$$

and from m squarefree, denote

$$\Omega_m = \bigcap_{\ell|m} \Omega_\ell.$$

We have to bound

$$\left| \bigcap_{\ell} (\Omega \setminus \Omega_\ell) \right| \leq \left| \bigcap_{\ell < Y_0} (\Omega \setminus \Omega_\ell) \right| \quad (6.3)$$

with $Y_0 < Y$ to be specified.

From the inclusion-exclusion principle

$$(6.3) \leq Y - \sum_{\ell < Y_0} |\Omega_\ell| + \sum_{\ell_1 < \ell_2 < Y_0} |\Omega_{\ell_1 \ell_2}| \dots + \sum_{\ell_1 < \dots < \ell_r < Y_0} |\Omega_{\ell_1 \dots \ell_r}| \quad (6.4)$$

with $r \in \mathbb{Z}_+$ even (to specify).

Clearly

$$|\Omega_m| = \left(\prod_{\ell|m} \frac{|R_\ell|}{\ell} \right) Y + o\left(\prod_{\ell|m} |R_\ell| \right). \quad (6.5)$$

From (6.4), (6.5)

$$\begin{aligned} \frac{(6.3)}{Y} &\leq 1 - \sum_{\ell < Y_0} \frac{|R_\ell|}{\ell} + \dots + \sum_{\ell_1 < \dots < \ell_r < Y_0} \frac{|R_{\ell_1}|}{\ell_1} \dots \frac{|R_{\ell_r}|}{\ell_r} \\ &\quad + \frac{1}{Y} \left(\sum_{\ell < Y_0} |R_\ell| + \dots + \sum_{\ell_1 < \dots < \ell_r < Y_0} |R_{\ell_1}| \dots |R_{\ell_r}| \right) \\ &\leq \prod_{\ell < Y_0} \left(1 - \frac{|R_\ell|}{\ell} \right) + \sum_{r_1 > r} \frac{1}{r_1!} \left(\sum_{\ell < Y_0} \frac{|R_\ell|}{\ell} \right)^{r_1} + \frac{2^{r+1}}{Y} \binom{Y_0 + r}{r} \\ &< \exp \left(3 \sum_{\substack{Y_0 < \ell < Y \\ \ell \text{ prime}}} \frac{1}{\ell} \right) \cdot \prod_{\ell < Y} \left(1 - \frac{|R_\ell|}{\ell} \right) + \sum_{r_1 > r} \left(\frac{2e \log \log Y}{r_1} \right)^{r_1} + (3Y_0)^r Y^{-1}. \end{aligned}$$

Take $r = 10^2 \log \log Y$, $Y_0 = Y^{10^{-3}(\log \log Y)^{-1}}$ to obtain (6.2).

Returning to Theorem 1 and alternative (4.2), we have

$$X \sim n = q + a = p_1 \dots p_r m \quad (\text{square free}) \quad (6.6)$$

where m has its prime factors in $\mathcal{P}(\mathcal{C}')$. Let $X' = \frac{X}{p_1 \dots p_r}$.

Thus if $\ell \notin \mathcal{P}(\mathcal{C}')$, $\pi_\ell(m) \neq 0$. Also, since q is prime, we have for any prime $\ell < \frac{X'}{4}$, $\ell \neq p_1, \dots, p_r$

$$\pi_\ell(m) \neq \pi_\ell(a)/\pi_\ell(p_1 \dots p_r).$$

Hence we define for $\ell \in \mathcal{P}(\mathcal{C}')$, $\ell < \frac{X'}{4}$, $\ell \neq p_1, \dots, p_r$

$$R_\ell = \{\pi_\ell(a)/\pi_\ell(p_1 \dots p_r)\},$$

and for $\ell \notin \mathcal{P}(\mathcal{C}')$, $\ell < \frac{X'}{4}$, $\ell \neq p_1, \dots, p_r$

$$R_\ell = \{0, \pi_\ell(a)/\pi_\ell(p_1 \dots p_r)\}$$

and $R_\ell = \emptyset$ otherwise.

Hence, recalling (5.3) and partial summation

$$\begin{aligned} \sum \frac{|R_\ell|}{\ell} &= \sum_{\substack{\ell \in \mathcal{P}(\mathcal{C}') \\ \ell < \frac{X'}{4}}} \frac{1}{\ell} + \sum_{\substack{\ell \notin \mathcal{P}(\mathcal{C}') \\ \ell < \frac{X'}{4}}} \frac{2}{\ell} + O(\log \log r) \\ &= 2 \log \log X' - \frac{1}{4} \log \log X' + o(\log \log X') \end{aligned} \quad (6.7)$$

(for $\log X' > (\log X)^{1/A}$).

Therefore, given $p_1 \dots p_r$, the number of possibilities for m in (6.6) is at most

$$\frac{X'}{(\log X')^{7/4-}} \quad (6.8)$$

using (6.2).

Assume $X' > X^\tau$. We obtain the bound (cf. (5.4))

$$\begin{aligned} \#\left\{ \mathcal{C}' < \mathcal{C}; [\mathcal{C} : \mathcal{C}'] \leq \frac{2}{\varepsilon} \right\} \frac{X}{\tau^{7/4}(\log X)^{7/4-}} \sum_{\substack{p_1 < \dots < p_r < X \\ \mathcal{X}_D(p_j) \neq -1, r < C(\varepsilon)}} \frac{1}{p_1 \dots p_r} \\ &< \#\left\{ \mathcal{C}' < \mathcal{C}; [\mathcal{C} : \mathcal{C}'] \leq \frac{2}{\varepsilon} \right\} \frac{X}{\tau^{7/4}(\log X)^{7/4-}} \frac{(\frac{\varepsilon}{2} \log \log X)^r}{r} \\ &\ll_\varepsilon \#\left\{ \mathcal{C}' < \mathcal{C}; [\mathcal{C} : \mathcal{C}'] \leq \frac{2}{\varepsilon} \right\} \cdot \frac{X}{\tau^{3/4}(\log X)^{7/4-}}. \end{aligned} \quad (6.9)$$

For $X' < X^\tau$, proceed as follows. Since $p_1 < \dots < p_r$ satisfies $p_1 \dots p_r > \sqrt{X}$, we have $p_r > X^{\frac{1}{2r}}$.

Writing

$$n = q + a = p_1 \dots p_{r-1} \cdot p_r \cdot m$$

and denoting $X'' = \frac{X}{p_1 \dots p_{r-1} m}$, fix p_1, \dots, p_{r-1}, m and estimate the number of possible $p_r \sim X''$. Thus, for primes $\ell < \frac{1}{4}X''$, $(\ell, p_1 \dots p_{r-1} m) = 1$,

$$\pi_\ell(p_r) \notin \{0, \pi_\ell(a)/\pi_\ell(p_1 \dots p_{r-1} m)\}$$

and, by Lemma 2, their number is at most

$$(\log \log X)^4 \frac{X''}{(\log X'')^2} < \frac{r^2 X''}{(\log X)^{2-}}. \quad (6.10)$$

This gives the contribution

$$\begin{aligned} & \sum_{\substack{p_1 \dots p_{r-1} m < X \\ m < X^\tau, m \text{ sf} \\ \mathcal{X}_D(p_1), \dots, \mathcal{X}_D(p_{r-1}) \neq -1 \\ \mathcal{X}_D(p) \neq -1 \text{ for } p|m}} \frac{X}{p_1 \dots p_{r-1} m} \cdot \frac{1}{(\log X)^{2-}} \\ & < \frac{X}{(\log X)^{2-}} \left(\frac{\frac{\varepsilon}{2} \log \log X}{r-1} \right)^{r-1} \left(\sum_{\substack{m < X^\tau \\ m \text{ sf, representable by } \mathcal{C}}} \frac{1}{m} \right) \\ & \stackrel{(5.11)}{<} \frac{X}{(\log X)^{2-}} \left(\frac{\frac{\varepsilon}{2} \log \log X}{r} \right)^r h^{1/A} (\log(X^\tau))^{\frac{1}{2}+} \\ & < \tau^{1/2} \frac{X}{(\log X)^{\frac{3}{2}-}}. \end{aligned} \quad (6.12)$$

Summing (6.9), (6.12) and appropriate choice of τ , gives the bound

$$\ll_\varepsilon \# \left\{ \mathcal{C}' < \mathcal{C}; [\mathcal{C} : \mathcal{C}'] \leq \frac{2}{\varepsilon} \right\} \frac{X}{(\log X)^{\frac{3}{2} + \frac{1}{20}}} \quad (6.13)$$

for the (4.2) exceptions.

Next the contribution of the (4.3) alternative from Theorem 1.

We have the bound (5.19), with the additional specification that $n = q + a$ (q prime) and recalling that $\Omega_{p_1 \dots p_k}$ only depends on the classes $C_1, \dots, C_k \in \mathcal{C}$ determined by p_1, \dots, p_k .

Write again

$$X \sim n = q + a = p_1 \dots p_{k-1} p_k m \text{ with } p_1 < \dots < p_k.$$

Assuming $p_1 \dots p_k < \sqrt{X}$, we fix p_1, \dots, p_k and observe that the number of possibilities for m with primes in $\mathcal{P}(\tilde{\Omega}_{p_1, \dots, p_k})$, is at most

$$\frac{X}{p_1 \dots p_{k-1} p_k} \cdot \frac{1}{(\log X)^{2-4\varepsilon}} \quad (6.14)$$

(using again Lemma 4).

This gives the contribution

$$\frac{X}{(\log X)^{2-4\varepsilon}} \sum_{\substack{p_1 < \dots < p_k \\ \mathcal{X}_D(p_j) \neq -1}} \frac{1}{p_1 \dots p_k} \quad (6.15)$$

with $k \leq (4.4)$. Following (5.22), (5.26), we get a bound

$$\frac{X}{(\log X)^{\frac{3}{2} + \frac{\varepsilon^2}{20}}} \quad (6.16)$$

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provided (5.24), i.e.

$$\frac{\log h}{\log \log X} < (1 - \kappa) \frac{\log 2}{2}. \quad (6.17)$$

If $p_1 \dots p_k \geq \sqrt{X}$, then $p_k > X^{\frac{1}{2k}}$.

Proceed as follows.

Fix p_1, \dots, p_{k-1} . Then specify the class $\{C, C^{-1}\}$ of (p_k) so that $\tilde{\Omega} = \tilde{\Omega}_{p_1, \dots, p_k}$ is specified. Take m with prime factors in $\mathcal{P}(\tilde{\Omega})$. Finally estimate the number of primes $p = p_k < \frac{X}{p_1 \dots p_{k-1} m}$ satisfying the condition

$$p \text{ represented by } C \quad (6.18)$$

$$\pi_\ell(p) \neq \pi_\ell(a)/\pi_\ell(p_1 \dots p_{k-1} m) \text{ if } (\ell, p_1 \dots p_{k-1} m) = 1, \ell < \sqrt{X}. \quad (6.19)$$

Lemma 5. *Let $Y < X$. Then*

$$|\{p < Y, p \text{ satisfies (6.18), (6.19)}\}| \ll \frac{(\log \log X)}{h^{1-\varepsilon}} \frac{Y}{(\log Y)^2}. \quad (6.20)$$

Thus we have the bound

$$\sum_{\substack{p_1 < \dots < p_{k-1} \\ \mathcal{X}_D(p_j) \neq -1}} \sum_{C \in \mathcal{C}} \sum_{\substack{m \text{ sf with primes in } \mathcal{P}(\tilde{\Omega}) \\ m < \frac{X^{1-\frac{1}{2k}}}{p_1 \dots p_{k-1}}}} \#\left\{p \lesssim \frac{X}{p_1 \dots p_{k-1} m}; (6.18), (6.19)\right\} \quad (6.21)$$

and applying Lemma 5.

$$\begin{aligned} \#\left\{p \lesssim \frac{X}{p_1 \dots p_{k-1} m}; (6.18), (6.19)\right\} &\ll \frac{k^2 (\log \log X)^4 X}{(\log X)^2 p_1 \dots p_{k-1} m h^{1-\varepsilon}} \\ &\stackrel{(4.4)}{\lesssim} \frac{(\log \log X)^6 X}{(\log X)^6 p_1 \dots p_{k-1} m h^{1-\varepsilon}}. \end{aligned} \quad (6.22)$$

Next, by (5.21),

$$\sum_{\substack{m < \frac{X}{p_1 \dots p_{k-1}} \\ \text{with primes in } \mathcal{P}(\tilde{\Omega})}} \frac{1}{m} \lesssim h^{1/A} (\log X)^{3\varepsilon}. \quad (6.23)$$

This gives the bound (after summation over $C \in \mathcal{C}$)

$$\begin{aligned} &\frac{h^{\varepsilon+1/A} X}{(\log X)^{2-4\varepsilon}} \sum_{\substack{p_1 < \dots < p_{k-1} < X \\ \mathcal{X}_D(p_j) \neq -1}} \left(\frac{1}{p_1 \dots p_{k-1}} \right) < \\ &\frac{X}{(\log X)^{2-5\varepsilon-\frac{1}{A}}} \left(\frac{\frac{\varepsilon}{2} \log \log X}{k-1} \right)^{k-1}. \end{aligned} \quad (6.24)$$

Using (6.24) and $k < \frac{1+\varepsilon}{\log 2} \log h$, the assumption (6.17) will again ensure (6.16).

Hence from (6.13), (6.16) and the preceding, we can conclude

Theorem 3. *Let $\kappa > 0$ be a fixed constant and $D < 0$, D not a perfect square s.t.*

$$|D| < (\log X)^{(1-\kappa)\log 2}. \quad (6.25)$$

Let \mathcal{C} be the class group. Then, for X large enough and $a \in \mathbb{Z}_+$, $a = o(X)$ fixed, we have
 $\#\{q + a \sim X; q \text{ prime, such that } q + a \text{ is squarefree and representable by some form but not by all forms of the genus}\} \lesssim_\kappa.$

$$\#\{\mathcal{C}' \text{ subgroup of } \mathcal{C}; [\mathcal{C} : \mathcal{C}'] < C(\kappa)\} \frac{X}{(\log X)^{\frac{3}{2} + \frac{1}{20}}} + \frac{X}{(\log X)^{\frac{3}{2} + \frac{\kappa^2}{20}}}. \quad (6.26)$$

Note that by decomposing \mathcal{C} into cyclic groups, one easily gets a bound

$$\#\{\mathcal{C}' \text{ subgroup of } \mathcal{C}; [\mathcal{C} : \mathcal{C}'] < C(\kappa)\} < C_1(\kappa)(\log h)^{C_1(\kappa)} < (\log \log X)^{C(\kappa)}.$$

Proof of Lemma 5.

In order to estimate the size of the set

$$\{p < Y, p \text{ satisfies (6.18), (6.19)}\} \quad (6.27)$$

we factor in prime ideals and consider the larger set

$$\{\alpha \in I; \alpha \in C, N(\alpha) < Y \text{ and } \pi_\ell(N(\alpha)) \notin R_\ell \text{ for } \ell < Y_0\} \quad (6.28)$$

where I denotes the integral ideals in O_K , $K = \mathbb{Q}(\sqrt{D})$, $D = D_0 f^2$ with $D_0 < 0$ squarefree, $N(\alpha)$ stands for the norm of α and ℓ runs over primes,

$$(6.29) \quad \begin{cases} R_\ell = \{0, \xi_\ell\}, \xi_\ell = \pi_\ell(a)/\pi_\ell(p_1 \dots p_{k-1}m) \text{ if } (\ell, p_1 \dots p_{k-1}m) = 1 \\ R_\ell = \{0\} \text{ otherwise.} \end{cases}$$

In fact, we restrict ourselves in (6.28) to primes $\ell < Y$ such that

$$(\ell, p_1 \dots p_{k-1}m) = 1. \quad (6.30)$$

Define

$$\Omega = \{\alpha \in I; \alpha \in C, N(\alpha) < Y\}$$

and

$$\Omega_\ell = \{\alpha \in \Omega; \pi_\ell(N(\alpha)) \in R_\ell\}$$

for ℓ prime,

$$\Omega_n = \bigcap_{\ell|n} \Omega_\ell$$

for n square-free.

Proceeding as in the proof of Lemma 4, estimate

$$\begin{aligned} & \left| \bigcap_{\ell < Y_0, (6.30)} (\Omega \setminus \Omega_\ell) \right| \leq \\ & |\Omega| - \sum_{\ell < Y_0, (6.30)} |\Omega_\ell| + \sum_{\ell_1 < \ell_2 < Y_0} |\Omega_{\ell_1, \ell_2}| - \dots + \sum_{\ell_1 < \dots < \ell_r < Y_0} |\Omega_{\ell_1, \dots, \ell_r}| \end{aligned} \quad (6.31)$$

with $r \in \mathbb{Z}_+$ $r \sim \log \log Y$ suitably chosen.

We evaluate $|\Omega_n|$ using Hecke characters.

The condition that $\alpha \in C$ becomes

$$\frac{1}{h} \sum_{\lambda \in \hat{C}} \overline{\lambda(C)} \lambda(\alpha) = 1 \quad (6.32)$$

where λ runs over the class group characters \hat{C} .

Denote \mathcal{X}_ℓ the principal character of $\mathbb{Q}(\text{mod } \ell)$.

If (6.30), $\pi_\ell(N(\alpha)) \in R_\ell$ may be expressed as

$$1 - \mathcal{X}_\ell(N(\alpha)) + \frac{1}{\ell-1} \sum_{\mathcal{X}(\text{mod } \ell)} \overline{\mathcal{X}(\xi_\ell)} \mathcal{X}(N(\alpha)) = 1. \quad (6.33)$$

Thus

$$|\Omega_n| = \sum_{N(\alpha) < Y} \left[\frac{1}{h} \sum_{\lambda \in \hat{C}} \overline{\lambda(C)} \lambda(\alpha) \right] \prod_{\ell|n} (6.33). \quad (6.34)$$

We will use the following classical extension of the Polya-Vinogradov inequality for finite order Hecke characters.

Proposition 6.

(i) Let \mathcal{X} be a non-principal finite order Hecke character (mod f) of K . Then

$$\left| \sum_{N(\alpha) < x} \mathcal{X}(\alpha) \right| < C(|D|N(f))^{1/3} [\log |D|N(f)]^2 x^{1/3} \quad (6.35)$$

and also

$$(ii) \quad \sum_{N(\alpha) < x} 1 = c_1 x + O(|D|^{1/3} (\log |D|)^2 x^{1/3}) \quad (6.36)$$

where

$$c_1 = \prod_{p|f} \left(1 - \frac{1}{p} \right) L(1, \mathcal{X}_D). \quad (6.37)$$

This statement follows from [L], (1), (2) p. 479; for (6.37), see [Bl], (2.5).

Analyzing (6.33), (6.34) more carefully, we see that

$$|\Omega_n| = \frac{1}{h} \sum_{N(\alpha) \leq Y} \prod_{\ell|n} \left(1 - \frac{\ell-2}{\ell-1} \mathcal{X}_\ell(N(\alpha)) \right) + O(6.39) \quad (6.38)$$

where (6.39) is a bound on sums

$$\sum_{N(\alpha) < Y} \mathcal{X}(\alpha) \text{ with } \mathcal{X}(\alpha) = \lambda(\alpha) \mathcal{X}'(N(\alpha)) \quad (6.40)$$

where $\lambda \in \hat{\mathcal{C}}$, \mathcal{X}' is a $(\bmod n_1)$ -Dirichlet character with $n_1|n$ and either λ or \mathcal{X}' non-principal. Thus by (6.35)

$$(6.39) < C|D|.nY^{1/3} < C|D|Y_0^rY^{1/3} \quad (6.41)$$

which collected contribution in (6.31) is at most

$$C|D|Y_0^{2r}Y^{1/3} < Y^{1/2} \quad (6.42)$$

imposing the condition

$$|D|Y_0^r < Y^{\frac{1}{20}}. \quad (6.43)$$

Analyzing further (6.37) using (6.36), we obtain

$$\begin{aligned} |\Omega_n| &= \frac{c_1}{h} \cdot Y \prod_{\substack{\ell|n \\ \mathcal{X}_D(\ell)=1}} \left[1 - \frac{\ell-2}{\ell-1} \left(1 - \frac{1}{\ell} \right)^2 \right] \cdot \prod_{\substack{\ell|n \\ \mathcal{X}_D(\ell)=0}} \left[1 - \frac{\ell-2}{\ell-1} \left(1 - \frac{1}{\ell} \right) \right] \\ &\quad \prod_{\substack{\ell|n \\ \mathcal{X}_D(\ell)=-1}} \left[1 - \frac{\ell-2}{\ell-1} \left(1 - \frac{1}{\ell^2} \right) \right] \\ &\quad + O(Y^{1/2}). \end{aligned} \quad (6.44)$$

Substituting (6.44) in (6.31) gives

$$\begin{aligned} &\frac{c_1}{h} Y \prod_{\substack{\ell < Y_0, (6.30) \\ \mathcal{X}_D(\ell)=1}} \left(1 - \frac{3}{\ell} + \frac{2}{\ell^2} \right) \cdot \prod_{\substack{\ell < Y_0, (6.30) \\ \mathcal{X}_D(\ell)=0}} \left(1 - \frac{2}{\ell} \right) \cdot \prod_{\substack{\ell < Y_0, (6.30) \\ \mathcal{X}_D(\ell)=-1}} \left(1 - \frac{1}{\ell} - \frac{2}{\ell^2} \right) + \\ &O\left(Y \frac{1}{r!} \left(\sum_{\substack{\ell < Y_0 \\ \ell \text{ prime}}} \frac{3}{\ell} \right)^r + Y^{1/2} Y_0^r \right) \\ &\ll \frac{|D|^\varepsilon}{h} Y \frac{(\log \log X)^3}{(\log Y_0)^2} + O\left(Y \left(\frac{3 \log \log Y_0}{r} \right)^r + Y^{1/2} Y_0^r \right). \end{aligned} \quad (6.45)$$

Taking $r = 10^2 \log \log Y$, $Y_0 = Y^{10^{-4}(\log \log Y)^{-1}}$, (6.43) holds and we obtain (6.20). This proves Lemma 5. \square

Theorem 3 may be combined with Iwaniec' result [I] on representing shifted primes by the genus of a binary quadratic form (see the Appendix for a quantitative review of that argument, when the quadratic form $Ax^2 + Bxy + Cy^2 = f(x, y)$ is not fixed). Thus, fixing $a \neq 0$, and assuming $D = B^2 - 4AC$ not a perfect square, it follows from [I] that

$$\begin{aligned} &\#\{q + a \sim X; q \text{ prime and } q + a \text{ squarefree and representable by the genus of } f\} \\ &\gg \frac{X}{(\log X)^{3/2+\varepsilon}} \end{aligned} \quad (6.46)$$

and this statement is certainly uniform assuming $|A|, |B|, |C| < \log X(?)$

Corollary 4. *Let f be as above with discriminant $D < 0$, and assume for some $\kappa > 0$*

$$|D| < (\log X)^{(1-\kappa) \log 2} \tag{6.47}$$

with X sufficiently large. Then

$$\begin{aligned} & \#\{q + a \sim X; q \text{ prime, such that } q + a \text{ is representable by } f\} \\ & \gg \frac{X}{(\log X)^{3/2+\varepsilon}}. \end{aligned}$$

Appendix

Let $\phi(x, y)$ be a primitive positive definite binary quadratic form of discriminant $-D$ where $D < \log X$, and let

$$S_1(X, \phi, a) = \sum_{\substack{p \leq X, p \nmid D \\ p \equiv f(x, y) + a \\ (x, y) = 1, f \in R_\phi}} 1$$

where R_ϕ denotes the genus of ϕ . Then Theorem 1 of [Iw] gives us the following lower bounds for S_1 .

Theorem A.1. *For $a \in \mathbb{Z}$ and ϕ a primitive positive definite binary quadratic form of discriminant $-D$ where $D \leq \log X$, let $S_1(X, \phi, a)$ be as above. Then for $\epsilon > 0$ we have*

$$S_1(X, \phi, a) \gg_\epsilon \frac{X \cdot D^{-\epsilon}}{(\log X)^{3/2}}$$

where the implied constant does not depend on D .

The following two lemmas are essentially Theorems 2 and 3 from [Iw] in the case $D < \log X$, where the integer m represented by R_ϕ is assumed to be square free and $(m, D) \leq 2$.

Lemma A.2. (Iwaniec). *Let $-D < 0$ be the discriminant of $f(x, y) = Ax^2 + 2Bxy + Cy^2$, and write*

$$-D = -2^{\theta_2} \cdot p_1^{\theta_{p_1}} \cdots p_r^{\theta_{p_r}}, \quad D_p = p^{-\theta_p} \cdot D,$$

where $\theta_p \geq 1$ for $1 \leq i \leq r$, and $\theta_2 \geq 0$. Write $m = \delta n = 2^{\epsilon_2} n$ where m is a positive square free integer (so $0 \leq \epsilon_2 \leq 1$) such that $(n, 2D) = 1$. Then m is represented by the genus of f iff the conditions on m in Table 1 are satisfied¹.

With the notation above, for $p \neq 2$, let

$$\mathcal{L}'_p(n) = \left\{ l \mid 0 < l < p, \left(\frac{l}{p} \right) = \left(\frac{A \cdot 2^{\epsilon_2}}{p} \right) \right\},$$

$$\mathcal{L}''_p(n) = \left\{ l \mid 0 < l < p, \left(\frac{l}{p} \right) = \left(\frac{-A \cdot 2^{\epsilon_2} \cdot k(-D_p)}{p} \right) \right\}$$

¹Table 1 also specifies a quantity κ and τ for each described case. These do not have to do with whether m is represented or not, but will be used later.

TABLE 1. Representation of $2^{\epsilon_2}n$ by f

Description of θ_p	\mathcal{K}	τ	Contributions on n	Contributions on D	
$\theta_{p_i} \geq 1, p_i \neq 2$	$\frac{p_i-1}{2}$	p_i	$\left(\frac{n}{p_i}\right) = \left(\frac{A \cdot 2^{\epsilon_2}}{p_i}\right)$	none	(1)
$p m, \theta_p = 0$	1	1	none	$\left(\frac{-D}{p}\right) = 1$	(2)
$\epsilon_2 = 0, \theta_2 = 0$	1	1	none	$D \equiv -1 \pmod{4}$	(3)
$\epsilon_2 = 0, \theta_2 = 2$	1 or 2	4	$n \equiv A \pmod{4}$ or $n \equiv -A D_2 \pmod{4}$	$D_2 \equiv -1 \pmod{4}$ or $D_2 \equiv 1 \pmod{4}$	(4)
$\epsilon_2 = 0, \theta_2 = 3$	2	8	$n \equiv A \pmod{8}$ or $n \equiv A(1 - 2D_2) \pmod{8}$	none	(5)
$\epsilon_2 = 0, \theta_2 = 4$	1	4	$n \equiv A \pmod{4}$	none	(6)
$\epsilon_2 = 0, \theta_2 \geq 5$	1	8	$n \equiv A \pmod{8}$	none	(7)
$\epsilon_2 = 1, \theta_2 = 0$	1	1	none	$D \equiv -1 \pmod{8}$	(8)
$\epsilon_2 = 1, \theta_2 = 2$	1	4	$n \equiv A \frac{1-D_2}{2} \pmod{4}$	$D_2 \equiv -1 \pmod{4}$	(9)
$\epsilon_2 = 1, \theta_2 = 3$	2	8	$n \equiv -A D_2 \pmod{8}$ or $n \equiv A(2 - D_2) \pmod{8}$	none	(10)

where $k(-D_p)$ denotes the square free kernel of $-D_p$. Note that each of \mathcal{L}'_p and \mathcal{L}''_p always contains $(p-1)/2$ elements. Define $\mathcal{L}_2(n)$ as follows:

$$\mathcal{L}_2(n) = \begin{cases} \{l \mid 0 < l < 4, l \equiv A \pmod{4} \text{ or } l \equiv -A D_2 \pmod{4}\} \text{ if } \epsilon_2 = 0, \theta_2 = 2 \\ \{l \mid 0 < l < 8, l \equiv A \pmod{8} \text{ or } l \equiv A(1 - 2D_2) \pmod{8}\} \text{ if } \epsilon_2 = 0, \theta_2 = 3 \\ \{l \mid 0 < l < 4, l \equiv A \pmod{4}\} \text{ if } \epsilon_2 = 0, \theta_2 = 4 \\ \{l \mid 0 < l < 8, l \equiv A \pmod{8}\} \text{ if } \epsilon_2 = 0, \theta_2 \geq 5 \\ \{l \mid 0 < l < 8, l \equiv -A D_2 \pmod{8} \text{ or } l \equiv A(2 - D_2) \pmod{8}\} \text{ if } \epsilon_2 = 1, \theta_2 = 3 \\ \{l \mid 0 < l < 4, l \equiv -A \frac{D_2 - 1}{2} \pmod{4}\} \text{ if } \epsilon_2 = 1, \theta_2 = 2, D_2 \equiv -1 \pmod{4} \\ \{0\} \text{ if } \epsilon_2 \geq \theta_2. \end{cases}$$

Note that $\mathcal{L}_2(n)$ contains κ elements, where κ is as in Table 1. With this notation, we have

Lemma A.3. (Iwaniec). *Let D, θ_p, m, n , and δ be as in Lemma 0.2, and let τ_2 be the corresponding value of τ in the case $p = 2$ in Table 1. Define $Q = \tau_2 \cdot \prod_{p_i | D_2} p_i$, and let*

$$P = \left\{ p \mid \left(\frac{k(-D)}{p} \right) = 1 \right\} \quad (\text{A.1})$$

where $k(-D)$ is the square free kernel of $-D$. Then $m = 2^{\epsilon_2}n$ is represented by the genus of ϕ iff m satisfies the conditions in Table 1, all the prime factors of n belong to P , and

$$n \equiv L \pmod{Q}$$

where $L > 0$ is an integer satisfying the conditions

- $0 < L < Q$,
- $L \equiv l \pmod{\tau_2}$ for some $l \in \mathcal{L}_2(n)$,
- for each $p_i | D_2$ there exists $l \in \mathcal{L}'_{p_i}(n)$ such that $L \equiv l \pmod{p_i}$.

Furthermore, if \mathcal{L} denotes the set of L satisfying these conditions, $\left(\frac{k(-D)}{L}\right) = 1$ for each $L \in \mathcal{L}$.

Let $\mathcal{P} = \{\text{primes } p \nmid D \text{ s.t. } \left(\frac{k(-D)}{p}\right) = -1\}$, let $E = Q\delta$, and let $\phi_E(N) = \phi(N \cdot E)/\phi(E)$. For D fixed, it is crucial to the $\frac{1}{2}$ -dimensional sieve that the condition

$$\left| \sum_{\substack{p \leq z \\ p \in \mathcal{P}}} \frac{\log p}{\phi_E(p)} - \frac{1}{2} \log z \right| < c \quad (\text{A.2})$$

is satisfied for some constant c for all $z > 1$. In our case of $D \leq \log X$, this holds in the following form for some constant C_1 not depending on D :

$$\left| \sum_{\substack{p \leq z \\ p \in \mathcal{P}}} \frac{\log p}{\phi_E(p)} - \frac{1}{2} \log z \right| \ll_{\epsilon} C_1 D^{\epsilon} \quad (\text{A.3})$$

for any $z \geq 1$. This can be seen from the proof of Theorem 3.2.1 of [Gl] and the fact that

$$\sum_{\left(\frac{k(-D)}{p}\right)=1, p \leq z} \frac{\log p}{p} = \frac{\log z}{2} + D^{\epsilon} \cdot O(1)$$

where the implied constant depends only on ϵ . As in [Iw], let

$$C_0 := \lim_{z \rightarrow \infty} \prod_{\substack{p < z \\ p \in \mathcal{P}}} \left(1 - \frac{1}{\phi_E(p)}\right) \sqrt{\log z}$$

for which Iwaniec shows in [Iw]

Lemma A.4. (Iwaniec). *Let C_0 be as above. We have*

$$C_0 = e^{-\gamma/2} \prod_{\substack{p \nmid a \\ p \in \mathcal{P}}} \left(1 - \frac{1}{(p-1)^2}\right) \cdot \prod_{p \mid Da} \left(1 - \frac{1}{p}\right)^{-1/2} \cdot \prod_{p \nmid Da} \left(1 - \frac{1}{p}\right)^{-\left(\frac{-k(D)}{p}\right)/2}$$

Finally, we recall the following theorem of Bombieri:

Lemma A.5. (Bombieri). *Let $\pi(x, k, l)$ denote the number of primes less than x which are l modulo k . There exists an absolute constant U such that*

$$\sum_{k < \frac{\sqrt{x}}{(\ln x)^U}} \max_{\substack{l \\ (l, k)=1}} \left| \pi(x, k, l) - \frac{\text{Li } x}{\phi(k)} \right| \ll \frac{x}{(\log x)^{20}}.$$

We are now ready to introduce the notation relevant to our problem and recall the lemmas resulting from the $\frac{1}{2}$ -dimensional sieve. For L and δ as above, and $1 < s \leq \frac{4}{3}$,

- $D_1 = 2$ or $1 =$ greatest divisor of $2D$ prime to $Q \cdot a$
- $M = \{m \in \mathbb{N} \mid m = \frac{p-a}{\delta}, p \leq X, p \equiv \delta L + a \pmod{Q\delta}, (m, D_1) = 1\}$
- $M_d = \{m \in M \mid m \equiv 0 \pmod{d}\}$
- $Y = \phi(E) \cdot |M| = \text{Li}(X)$
- $R_d(M) = |M_d| - \frac{Y}{\phi(dE)}$
- $y = \frac{\sqrt{X}}{Q\delta D(\log X)^\sigma}$
- $A(M, y^{1/s}) = \#\{m \in M \text{ s.t. } m \not\equiv 0 \pmod{p}, y^{1/s} > p \in \mathcal{P}\}$

By Lemma A.3, the following is precisely what is needed to evaluate S_1 :

$$\begin{aligned} \sum_{\substack{|a| < f(x,y) + a = p \leq X \\ (x,y)=1, f \in R_\phi}} 1 &= \sum_d \sum_{L \in \mathcal{L}} \sum_{\substack{X \geq p \equiv \delta L + a \pmod{Q\delta} \\ q \mid ((p-a)/\delta) \Rightarrow q \in P \\ ((p-a)/\delta, 2D)=1, p > |a|}} 1 \\ &= \sum_{\substack{\delta \\ 2 \mid a\delta}} \sum_{\substack{L \in \mathcal{L} \\ (\delta L + a, Q\delta)=1}} \sum_{\substack{m \in M \\ q \mid m \Rightarrow q \in P}} 1 + \mathcal{R} \end{aligned} \quad (\text{A.5})$$

where $\mathcal{R} \leq 2|D|$. It is the innermost sum in (A.5) that we evaluate with the help of the sieve. Note that if $(d, QA) = 1$, there exists an integer d' such that $d'Q + L \equiv 0 \pmod{d}$ and

$$M_d = \{m \mid m = \frac{p+a}{\delta}, p \leq X, p \equiv A + \delta L + Q\delta d' \pmod{Q\delta d}\}.$$

From [Iw] we then have

$$\left| |M_d| - \frac{\text{Li} X}{\phi(Q\delta d)} \right| \leq 2 \max_{(l, Q\delta d)=1} \left| \pi(X, Q\delta d, l) - \frac{\text{Li} X}{\phi(Q\delta d)} \right| \quad (\text{A.6})$$

With the notation above, the expression in (A.3) combined with the $\frac{1}{2}$ -dimensional sieve gives the following in our case:

Theorem A.6.

$$\begin{aligned} A(M, y^{1/s}) &\gg_\epsilon \sqrt{\frac{e^\gamma}{\pi}} \cdot \frac{C_0 Y}{\phi(E) \sqrt{\log 3y}} \cdot \left(\int_1^s \frac{dt}{\sqrt{t(t-1)}} - \frac{(\log X)^\epsilon}{(\log 3y)^{1/10}} \right) - \sum_{\substack{d < y \\ p \mid d \Rightarrow p \in \mathcal{P}}} |R_d(M)| \\ &\geq \frac{C_0 \cdot \sqrt{2 \frac{e^\gamma}{\pi}}}{\phi(Q\delta)} \cdot \frac{X}{(\log X)^{3/2}} \cdot \left(\int_1^s \frac{dt}{\sqrt{t(t-1)}} + (\log X)^\epsilon \cdot o(1) \right) + O(X \log^{-20} X). \end{aligned}$$

The estimation of the remainder term comes from Lemma A.5 and (A.6). Also, for sufficiently large X (such that $(X^{1/2}(\log X)^{-15-U})^{1/s} > X^{1/3}$) and $1 < s < \frac{4}{3}$ we have

$$A(M, y^{1/s}) = \sum_{\substack{m \in M \\ q \mid m \Rightarrow q \in P}} 1 + \sum_{\substack{p_1 p_2 m \in M \\ q \mid m \Rightarrow q \in P \\ y^{1/s} \leq p_1, p_2 \in \mathcal{P}}} 1.$$

We have a lower bound for $A(M, y^{1/s})$, and we would like a lower bound for the first sum in the equation above. To this end, Iwaniec shows:

Lemma A.7. (Iwaniec). *Let $|Q\delta| \ll (\log X)^{15}$ and $s > 1$. Then*

$$\sum_{\substack{p_1 p_2 m \in M \\ q|m \Rightarrow q \in P \\ y^{1/s} \leq p_1, p_2 \in P}} 1 < \frac{4e^{\gamma/2} C_0 \sqrt{s-1}}{\sqrt{\pi} \phi(Q\delta) \sqrt{s}} \log(2s-1) \frac{4s^2 X}{(\log X)^{3/2}} (1 + o(1))$$

Together with Theorem A.6, for $1 < s < \frac{4}{3}$ and $Q\delta \ll \log X$, this gives us

$$\sum_{\substack{m \in M \\ q|m \Rightarrow q \in P}} 1 \gg \sqrt{\frac{2e^\gamma}{\pi}} \cdot \frac{C_0}{\phi(Q\delta)} \cdot \frac{X}{(\log X)^{3/2}} \cdot \left(\int_1^s \frac{dt}{\sqrt{t(t-1)}} - 8s^2 \sqrt{2 \frac{s-1}{s}} \log(2s-1) + o(1) \right) + O(X \log^{-20} X),$$

where the implied constants do not depend on D .

We now compute a lower bound for the expression in (A.5) as in [Iw]. Since D_1 in our case is 1 or 2, the expression Ω_D in (4.8) of [Iw] becomes

$$\Omega_D = c \cdot \sum_{\substack{\delta \\ 2|D\delta}} \sum_{\substack{L \in \mathcal{L} \\ (\delta L + a, Q\delta)=1}} \frac{1}{\phi(Q\delta)} \quad (\text{A.7})$$

where c is a constant not depending on D (coming from the products over $p|D_1$ in (4.8) of [Iw]) and $\delta = 1$ or 2 as in Table 1. Note that the innermost sum of the expression in (4.8) is $\gg_\epsilon D^{-\epsilon}$ for $\epsilon > 0$. This follows from $|\mathcal{L}| = \prod_{p|D_2} (p-1)/2 \gg_{\epsilon'} D^{1-\epsilon'}$. Define

$$\tilde{\Omega}_D = \sum_{\substack{\delta \\ |Q\delta| \leq \log^{15} X \\ 2|D\delta}} \sum_{\substack{L \in \mathcal{L} \\ (\delta L + a, Q\delta)=1}} \frac{1}{\phi(Q\delta)}$$

and note that, since $\delta \leq 2$ and $D \leq \log X$ in our case,

$$\begin{aligned} |\Omega_a - \tilde{\Omega}_a| &\leq \sum_{\substack{\delta \\ Q\delta > \log^{15} X \\ p|\delta \Rightarrow p|D}} \frac{Q}{\phi(Q\delta)} < |8D| \cdot \sum_{Q\delta > \log^{15} X} \frac{1}{\sqrt{Q\delta} \sqrt{\phi(Q)}} \\ &< \frac{|8D|}{\log^{7.5} X} \\ &\leq \frac{1}{\log^6 X} \end{aligned}$$

Combined with Theorem 1 of [Iw], this gives us the following bounds for $S_1(\phi, X, a)$ where $D \leq \log X$ and $\delta = 1$ or 2 :

$$\begin{aligned} S_1 &\geq \theta \sqrt{\frac{2e^\gamma}{\pi}} C_0 \cdot \tilde{\Omega}_a \frac{X}{(\log X)^{3/2}} (1 + o(1)) + O(X \log^{-20} X) \\ &= \theta \Psi_D \Omega_D \frac{X}{(\log X)^{3/2}} (1 + o(1)) + O(X \log^{-6} X) \end{aligned}$$

where the implied constants do not depend on D ,

$$\theta = \sup_{1 < s < 4/3} \left(\int_1^s \frac{dt}{\sqrt{t(t-1)}} - 8s^2 \sqrt{\frac{2(s-1)}{s}} \log(2s-1) \right),$$

$$C_0 = \Psi_D = \sqrt{\frac{2}{\pi}} \prod_{p|2Da} \left(1 - \frac{1}{p}\right)^{-1/2} \prod_{\substack{p \nmid 2Da \\ \left(\frac{k(-D)}{p}\right) = -1}} \left(1 - \frac{1}{(p-1)^2}\right) \prod_{p \nmid 2Da} \left(1 - \frac{1}{p}\right)^{-\frac{1}{2}\left(\frac{k(-D)}{p}\right)} \gg_{\epsilon} D^{-\epsilon}$$

and $\Omega_D \gg_{\epsilon} D^{-\epsilon}$ as well for $\epsilon > 0$. This gives us the desired generalization of Iwaniec's theorem to Theorem A.1.

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